Mathematical Preliminaries

Developed for the ERCOT Synchronization Project

By: Joseph D. Fournier B.Sc.E.E., M.Sc.E.E.

Outline

Sets

Developed for the Members of Azera Global

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Outline

Introduction: Part One

- We have all implicitly dealt with sets
  - Integers ($\mathbb{Z}$), rationals ($\mathbb{Q}$), naturals ($\mathbb{N}$), reals ($\mathbb{R}$), etc.
- We will develop more fully
  - The definitions of sets
  - The properties of sets
  - The operations on sets
- **Definition**: A set is an <u>unordered</u> collection of (<u>unique</u>) objects
- Sets are fundamental discrete structures and for the basis of more complex discrete structures like graphs

Introduction: Part Two

- The objects in a set are called elements or members of a set. A set is said to contain its elements
- Notation, for a set A:
  - $x \in A$: x is an element of A
  - $x \notin A$: x is not an element of A

Properties: Part One

- Two sets, A and B, are equal is they contain the same elements. We write A=B.
- Example:
  - {2,3,5,7}={3,2,7,5}, because a set is unordered
  - Also, {2,3,5,7}={2,2,3,5,3,7} because a set contains unique elements
  - However, {2,3,5,7} $\neq${2,3}

Properties: Part Two

- A multi-set is a set where you specify the number of occurrences of each element: $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ is a set where
  - m_1 occurs a_1 times
  - m_2 occurs a_2 times
  - …
  - m_r occurs a_r times
- In Databases, we distinguish
  - A set: elements cannot be repeated
  - A bag: elements can be repeated

Terminology

- The **set-builder** notation

  $O=\{ x \mid (x \in \mathbb{Z}) \wedge (x=2k) \text{ for some } k \in Z\}$

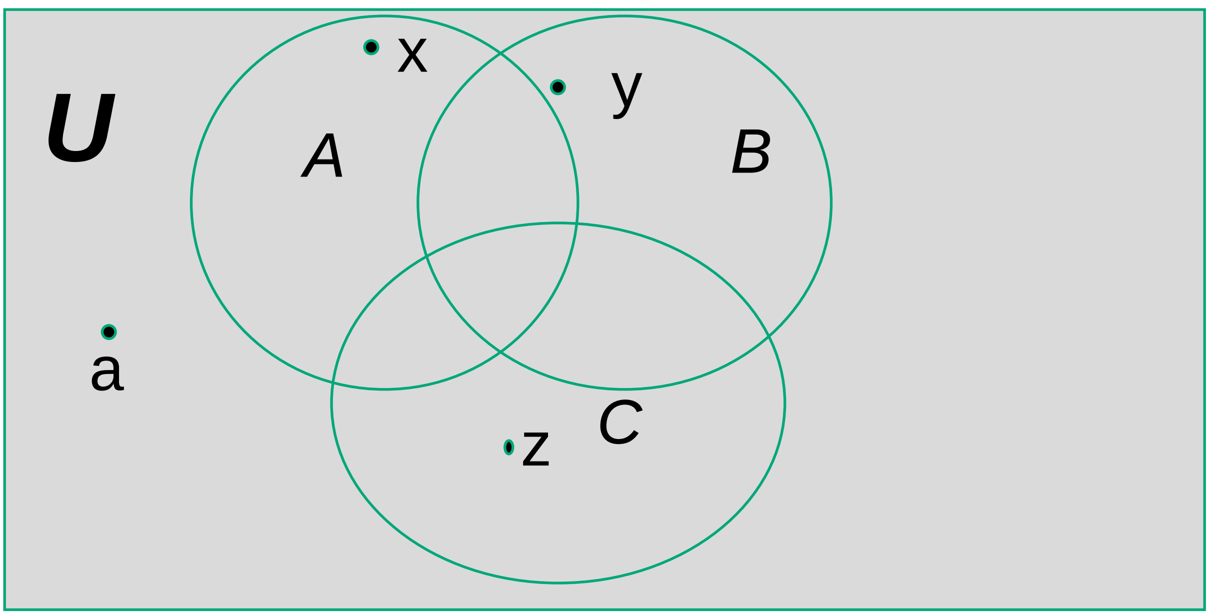
  reads: O is the set that contains all x such that x is an integer and x is even
- A set is defined in **intension** when you give its set-builder notation

  $O=\{ x \mid (x \in \mathbb{Z}) \wedge (0 \leq x \leq 8) \wedge (x=2k) \text{ for some } k \in Z \}$
- A set is defined in **extension** when you enumerate all the elements:

  $O=\{0,2,4,6,8\}$

Venn Diagram:

- A set can be represented graphically using a Venn Diagram



Properties and Notation: Part One

- A set that has no elements is called the **empty set** or **null set** and is denoted $\varnothing$
- A set that has one element is called a **singleton set**.
  - For example: {a}, with brackets, is a singleton set
  - a, without brackets, is an element of the set {a}
- Note the subtlety in $\varnothing \neq \{\varnothing\}$
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set

Properties and Notation: Part Two

- For any set S
  - $\emptyset \subseteq S$ and
  - $S \subseteq S$
- A is said to be a **subset** of B, and we write $A \subseteq B$, if and only if every element of A is also an element of B
- That is, we have the equivalence:

$$A \subseteq B \Leftrightarrow \forall x \, (x \in A \Rightarrow x \in B)$$

Properties and Notation: Part Three

- A set A that is a subset of a set B is called a **proper subset** if $A \neq B$.
- That is there is an element $x \in B$ such that $x \notin A$
- We write: $A \subset B$,

If there are exactly n distinct elements in a set S, with n a nonnegative integer, we say that:

S is a finite set, and

The cardinality of S is n. Notation: $|S| = n$.

A set that is not finite is said to be infinite

Equivalence: Part One

- To show that a set is
  - a subset of,
  - proper subset of, or
  - equal to another set.
- To prove that A is a subset of B, use the equivalence discussed earlier $A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$
  - To prove that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element x, $x \in A$ implies that x is also in B.
- To prove that A is a proper subset of B, you must prove
  - A is a subset of B **and**
  - $\exists x\ (x \in B) \wedge (x \notin A)$

Equivalence: Part Two

- To show that two sets are equal, it is sufficient to show independently (much like a biconditional) that
  - $A \subseteq B$ and
  - $B \subseteq A$
- Logically speaking, you must show the following quantified statements:

$$(\forall x\ (x \in A \Rightarrow x \in B)) \wedge (\forall x\ (x \in B \Rightarrow x \in A))$$

Power Set

- The power set of a set S, denoted P(S), is the set of all subsets of S.
- Examples
  - Let A={a,b,c}, P(A)={∅,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}
  - Let A={{a,b},c}, P(A)={∅,{{a,b}},{c},{{a,b},c}}
- Note: the empty set ∅ and the set itself are always elements of the power set.
- The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set
- Let S be a set such that |S|=n, then

$$|P(S)| = 2^n$$

Tuples

- Sometimes we need to consider ordered collections of objects
- The ordered n-tuple $(a_1,a_2,...,a_n)$ is the ordered collection with the element a_i being the i-th element for i=1,2,...,n
- A 2-tuple (n=2) is called an ordered pair

Cartesian Product

- Let A and B be two sets. The Cartesian product of A and B, denoted AxB, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$

$$AxB = \{ (a,b) \mid (a \in A) \wedge (b \in B) \}$$

- The Cartesian product is also known as the cross product
- A subset of a Cartesian product, $R \subseteq AxB$ is called a relation.
- Note: $AxB \neq BxA$ unless $A=\varnothing$ or $B=\varnothing$ or $A=B$
- Cartesian Products can be generalized for any n-tuple
- The Cartesian product of n sets, $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is

$$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i=1,2,\ldots,n \}$$

Notation with Quantifiers

- Whenever we wrote $\exists x P(x)$ or $\forall x P(x)$, we specified the universe of discourse using explicit English language
- Now we can simplify things using set notation!
- Example
  - $\forall\ x \in \mathcal{R}\ (x^2 \geq 0)$
  - $\exists\ x \in \mathcal{Z}\ (x^2 = 1)$
  - Also mixing quantifiers:

$$\forall a,b,c \in \mathcal{R}\ \exists\ x \in \mathcal{C}\ (ax^2+bx+c=0)$$

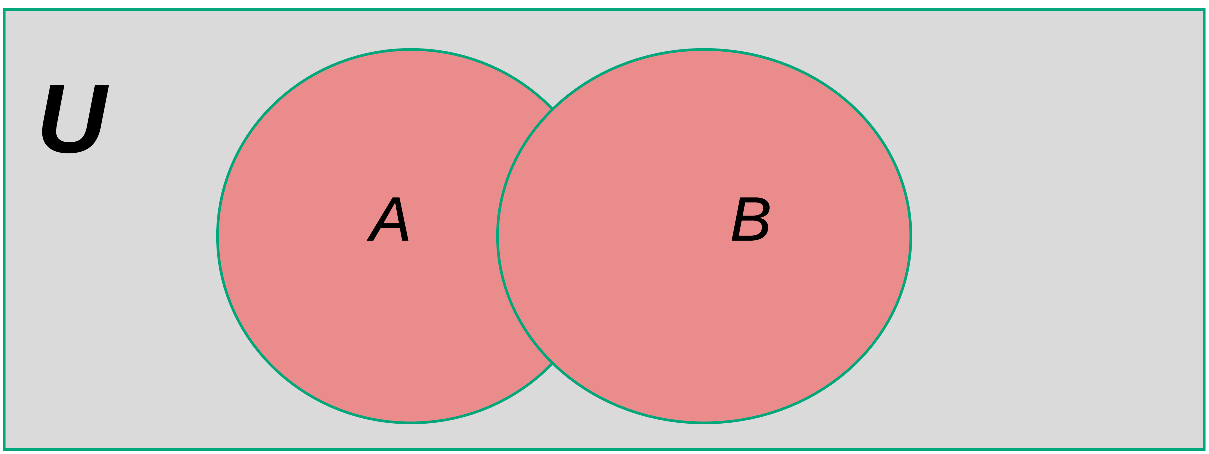
Set Operations

- Arithmetic operators (+,-, × ,÷) and set operators exist and act on two sets to give us new sets
  - Union
  - Intersection
  - Set difference
  - Set complement
  - Generalized union
  - Generalized intersection

Set Operators: Union

- The union of two sets A and B is the set that contains all elements in A, B, r both. We write:

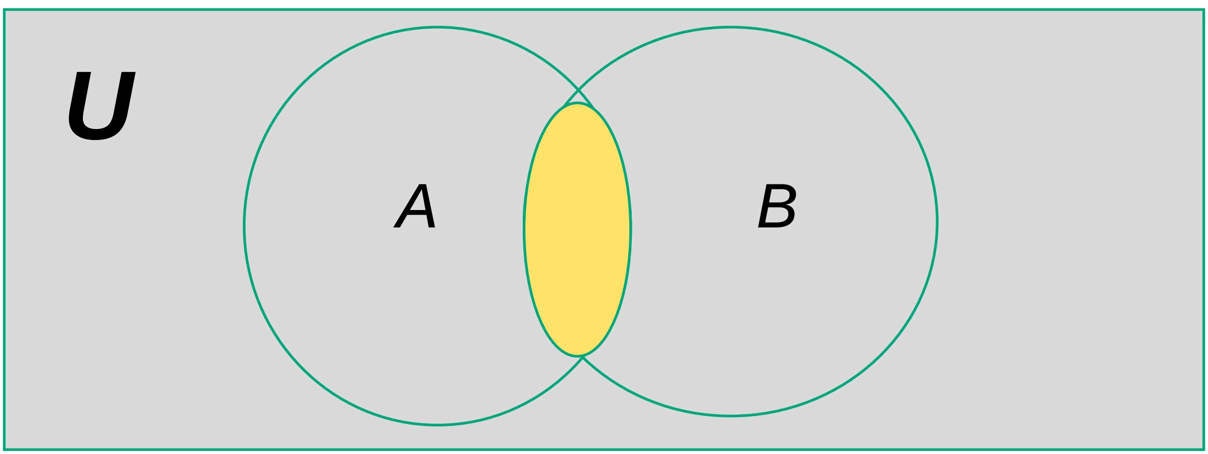
$$A \cup B = \{ x \mid (x \in A) \vee (x \in B) \}$$



Set Operators: Intersection

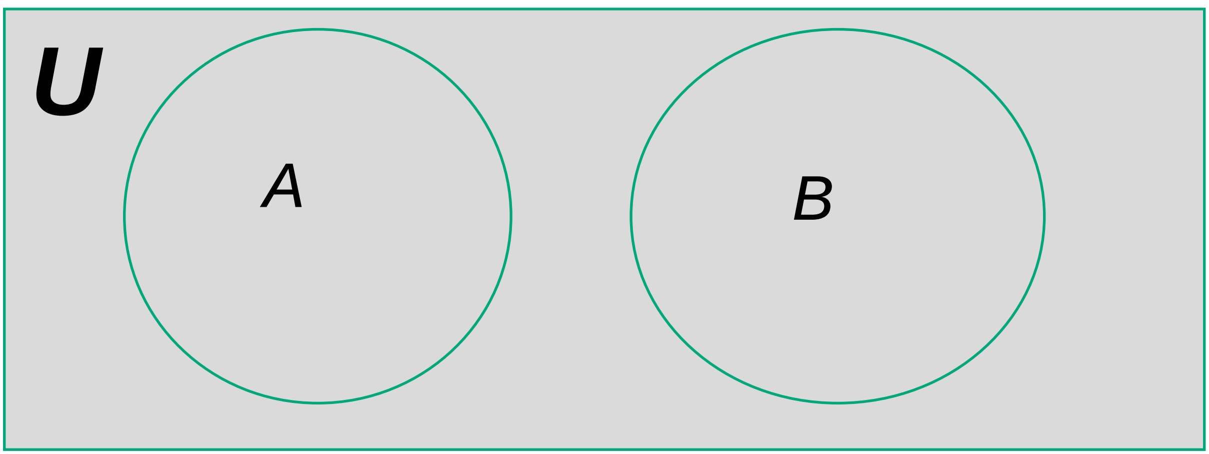
- The intersection of two sets A and B is the set that contains all elements that are element of both A and B. We write:

$$A \cap B = \{ x \mid (x \in A) \wedge (x \in B) \}$$



Disjoint Sets

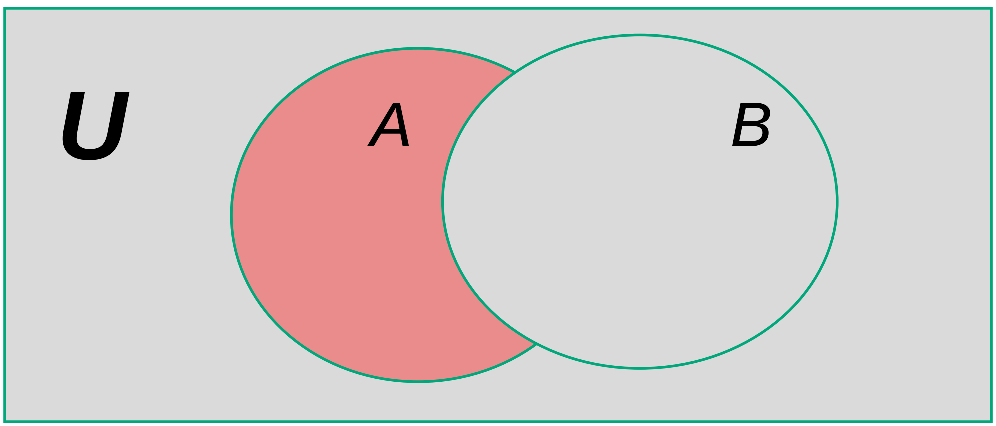
- Two sets are said to be disjoint if their intersection is the empty set: $A \cap B = \varnothing$



Set Difference

- The difference of two sets A and B, denoted A\B or A−B, is the set containing those elements that are in A but not in B

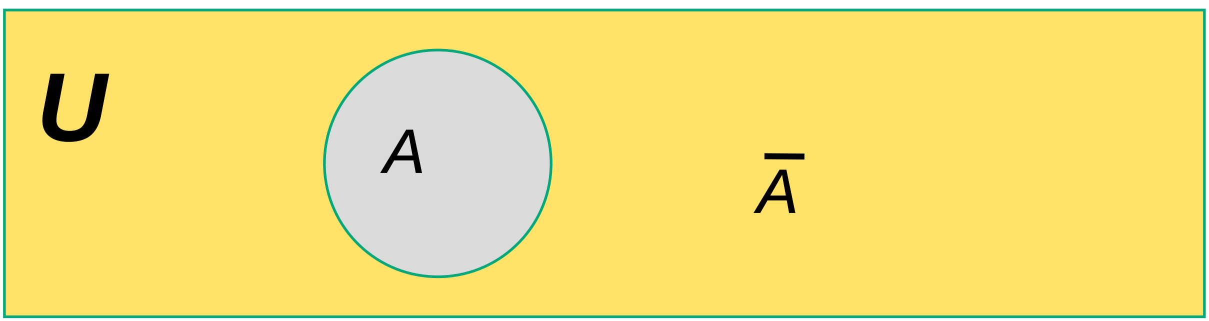
$$A - B = \{ x \mid (x \in A) \wedge (x \notin B) \}$$



Set Complement

- **Definition**: The complement of a set A, denoted $\overline{A}$, consists of all elements <u>not</u> in A. That is the difference of the universal set and U: U\A

$$A = \overline{A} = \{x \mid x \notin A\}$$



Generalized Union

- The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

Generalized Intersection

- The intersection of a collection of sets is the set that contains those elements that are members of every set in the collection

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Chapter Two: Introduction to Functions

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Outline

Relations and Functions

Relation

A **relation** is any set of ordered pairs.

A special kind of relation, called a *function*, is very important in mathematics and its applications.

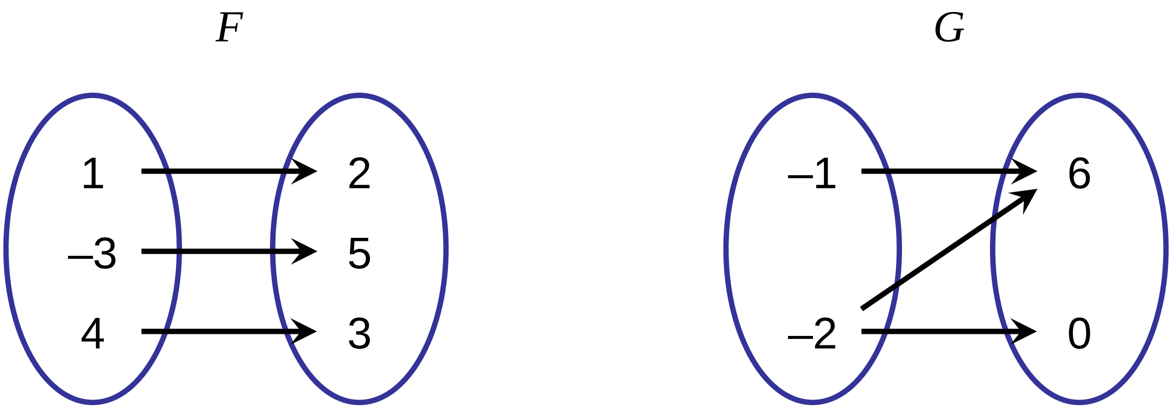
Function

A **function** is a relation in which, for each value of the first component of the ordered pairs, there is *exactly one value* of the second component.

In a relation, the set of all values of the independent variable (x) is the **domain**.

The set of all values of the dependent variable (y) is the **range**

Introduction to Functions



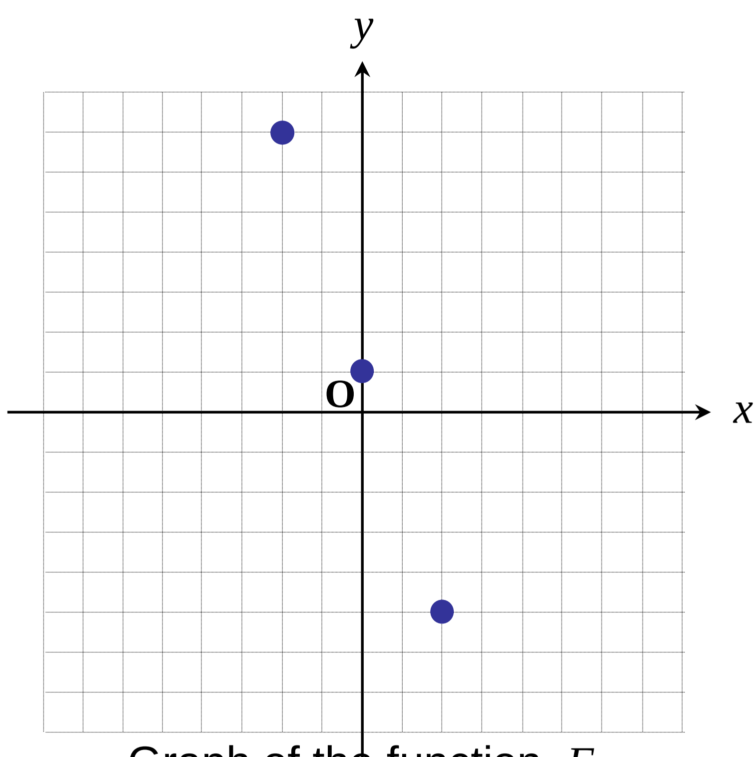
F is a function.

G is not a function.

Tables and Graphs

x	y
–2	6
0	0
2	–6

Table of the function, F



Graph of the function, F

Function Notation

When a function f is defined with a rule or an equation using x and y for the independent and dependent variables, we say "y is a function of x" to emphasize that y *depends on* x. We use the notation

$$y = f(x),$$

called **function notation**, to express this and read $f(x)$, as "f of x".

The letter f stands for *function*. For example, if $y = 5x - 2$, we can name this function f and write

$$f(x) = 5x - 2.$$

Note that $f(x)$ ***is just another name for the dependent variable*** y**.**

Linear Function

A function that can be defined by

$$f(x) = ax + b,$$

for real numbers a and b is a **linear function.**

The value *of a* is the slope of *m* of the graph of the function. Before we can draw a graph of our function we must look at the co-ordinate plane or the Cartesian Co-ordinate plane.

The Co-ordinate Plane

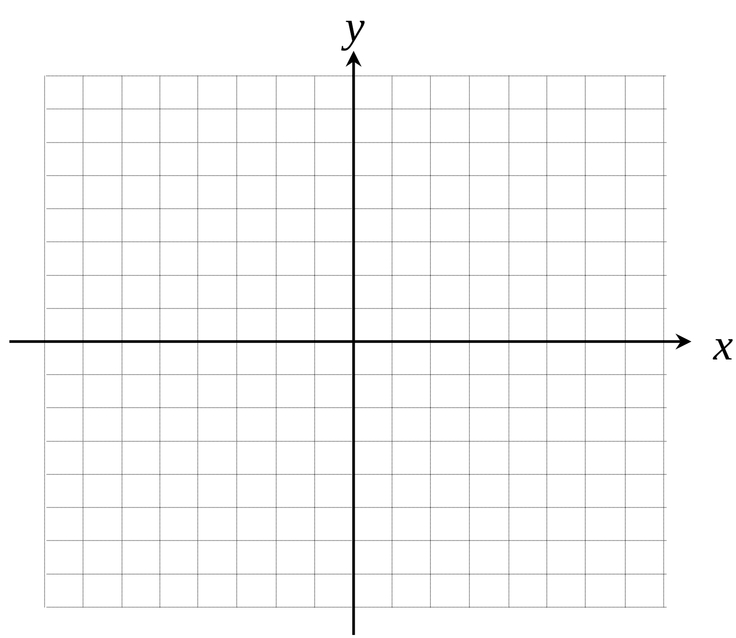
A function that can be defined by $f(x) = ax + b,$

The plane of the grid is called the coordinate plane.

The horizontal number line is called the _x-axis_.

The vertical number line is called the _y-axis_.

The point of intersection of the two axes is called the origin



Graphing a Function

An ordered pair of real numbers, called coordinates of a point, locates a point in the coordinate plane.

Each ordered pair corresponds to EXACTLY <u>one point</u> in the coordinate plane.

The point in the coordinate plane is called the graph of the ordered pair.

Locating a point on the coordinate plane is called <u>graphing</u> the ordered pair.

Chapter Three:
Logarithmic Functions

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Outline

- Slide 39-Definition: Logarithmic Functions
- Slide 40-Properties of Logarithms
- Slide 41-Properties of Natural Logarithms
- Slide 42-Properties of Natural Logarithms
- Slide 43-Characteristics of $f(x)=\log_b x$
- Slide 44-Domain of Logarithmic Functions

Definition: Logarithmic Function

For x > 0 and b > 0, b = 1,

$$y = \log_b x \text{ is equivalent to } b^y = x.$$

The function $f(x) = \log_b x$ is the logarithmic function with base b.

Exponent | Exponent

Logarithmic form: $\boldsymbol{y} = \log_b \boldsymbol{x}$ Exponential Form: $b^{\boldsymbol{y}} = \boldsymbol{x}$.

Base | Base

Properties of Logarithms

For x > 0 and b $\neq$ 1,

- $\log_b b^x = \boldsymbol{x}$ The logarithm with base b of b raised to a power equals that power.
- $b^{\log_b x} = \boldsymbol{x}$ b raised to the logarithm with base b of a number equals that number.

General Properties: Common Logarithms

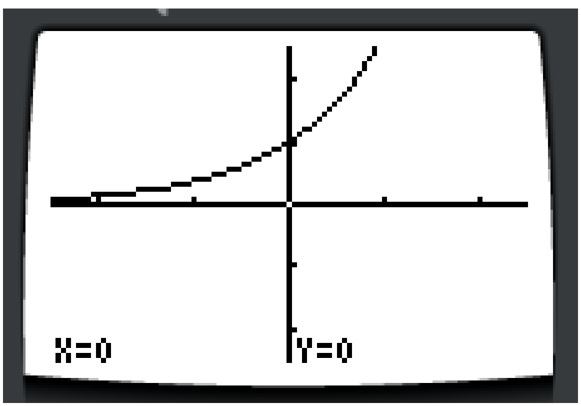
General Properties	Common Logarithms
1. $\log_b 1 = 0$	1. $\log 1 = 0$
2. $\log_b b = 1$	2. $\log 10 = 1$
3. $\log_b b^x = 0$	3. $\log 10^x = \boldsymbol{x}$
4. $b^{\log_b x} = \boldsymbol{x}$	4. $10^{\log x} = \boldsymbol{x}$

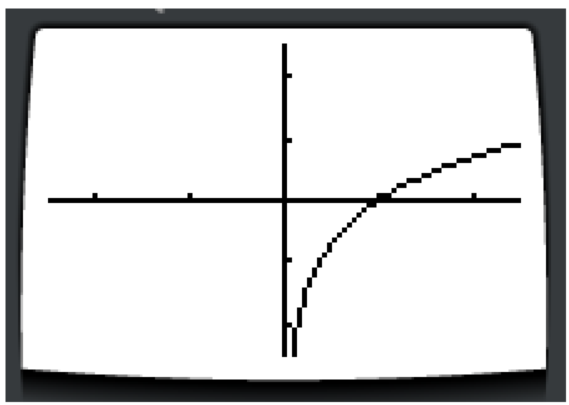
Properties of Natural Logarithms

General Properties	**Natural Logarithms**
1. $\log_b 1 = 0$	1. $\ln 1 = 0$
2. $\log_b b = 1$	2. $\ln e = 1$
3. $\log_b b^x = 0$	3. $\ln e^x = \boldsymbol{x}$
4. $b^{\log_b x} = \boldsymbol{x}$	4. $e^{\ln x} = \boldsymbol{x}$

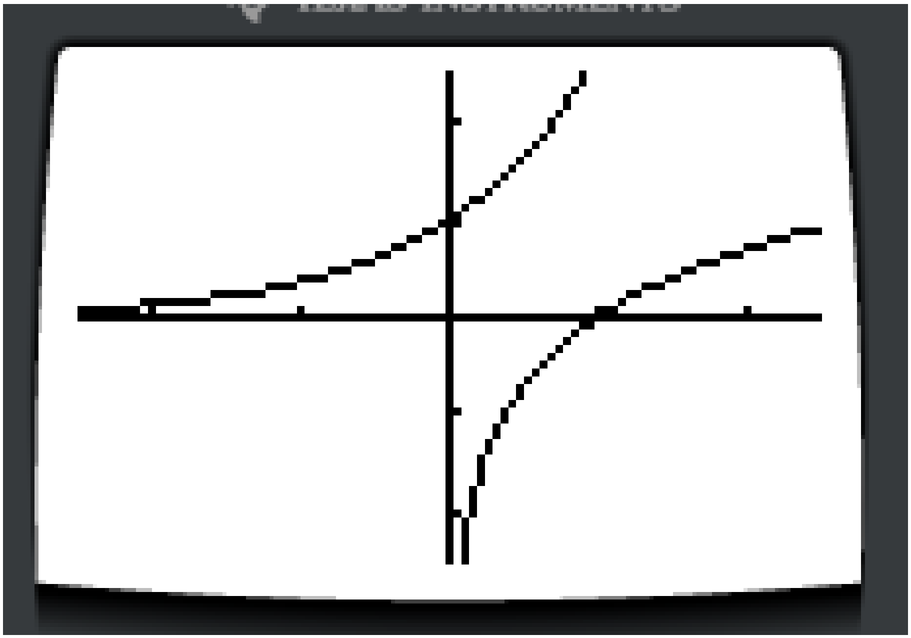
The function $y=e^x$ has an inverse called the Natural Logarithmic Function.

Y=ln x





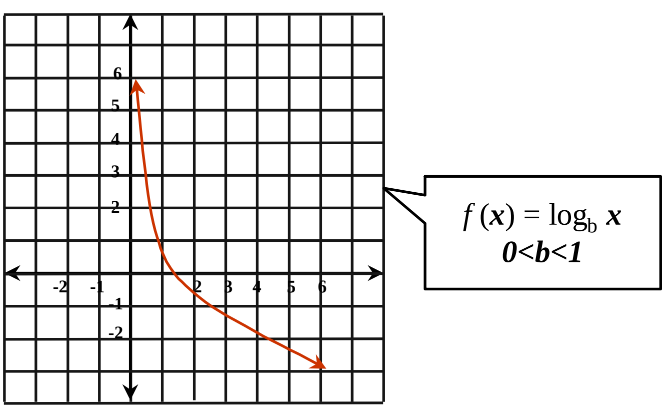
Properties of Natural Logarithms

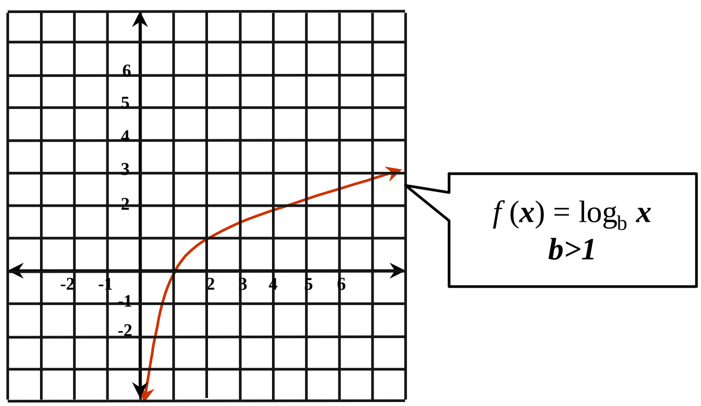


$y=e^x$ and $y=\ln x$ are inverses of each other!

Characteristics of $f(x) = \log_b x$

- The x-intercept is 1. There is no y-intercept.
- The y-axis is a vertical asymptote. (x = 0)
- If 0 < b < 1, the function is decreasing. If b > 1, the function is increasing.
- The graph is smooth and continuous. It has no sharp corners or edges.





Domain of Logarithmic Functions

Because the logarithmic function is the inverse of the exponential function, its domain and range are the reversed. $f(x) = \log_b (x + c)$

The domain is { x | $x > 0$ } and the range will be all real numbers.

For variations of the basic graph, say the domain will consist of all x for which $x + c > 0$.

Chapter Four: Trigonometry

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Outline

- Slide 47-Right Triangle Trigonometry
- Slide 48-Right Triangle Trigonometry
- Slide 49-Trigonometric Ratios
- Slide 50-Reciprocal Functions
- Slide 51-Important Trigonometric Identities

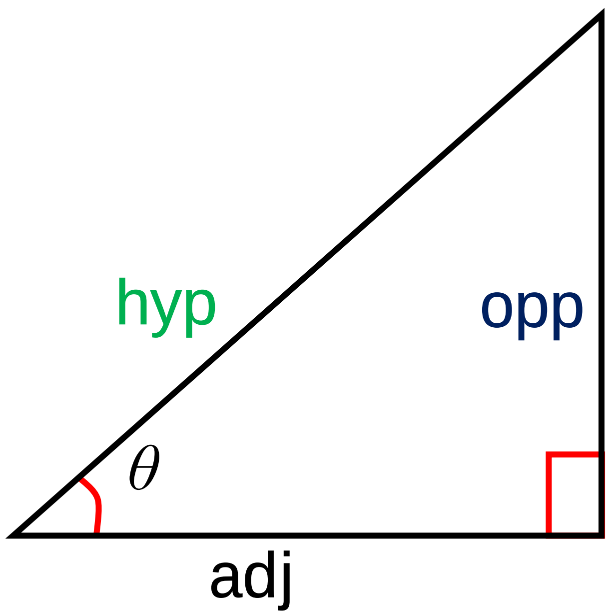
Right Triangle Trigonometry

Trigonometry is based upon ratios of the sides of right triangles.

The six **trigonometric functions** of a right triangle, with an acute angle, are defined by **ratios** of two sides of the triangle.

The sides of the right triangle are:

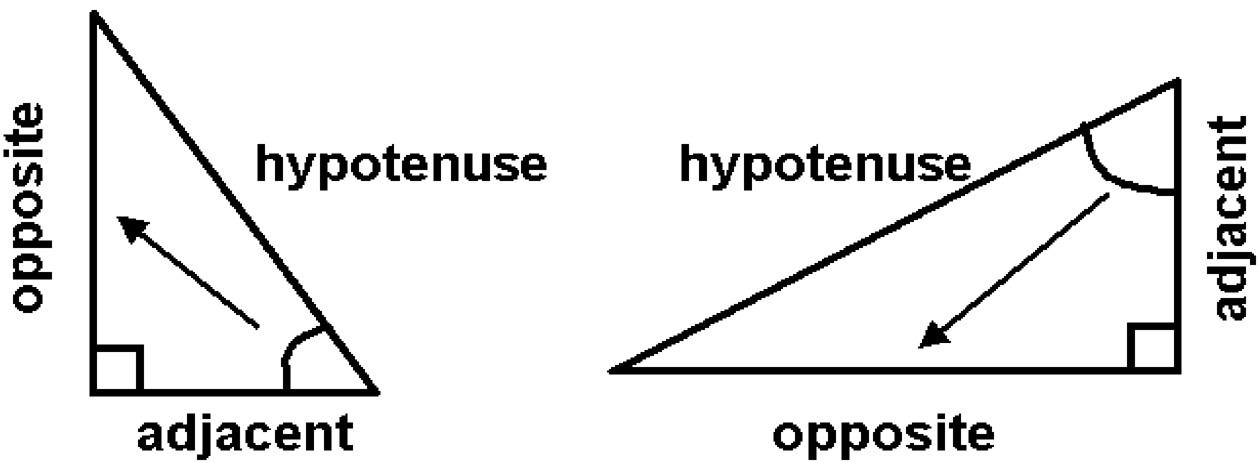
- **opposite**
- **adjacent**
- **hypotenuse**



Right Triangle Trigonometry

The hypotenuse is the longest side and is always opposite the right angle.

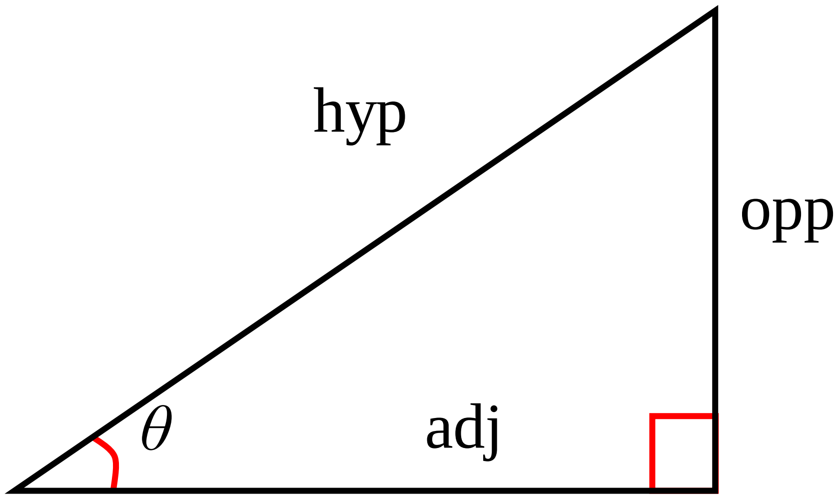
The opposite and adjacent sides refer to another angle, other than the 90º.



Trigonometric Functions

sine, cosine, tangent,

cotangent, secant, and cosecant.



$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp} \qquad \tan\theta = \frac{opp}{adj}$$

$$\csc = \frac{hyp}{opp} \qquad \sec = \frac{hyp}{adj} \qquad \cot = \frac{adj}{opp}$$

Reciprocal Functions

$\sin\theta = 1/\csc\theta$

$\cos\theta = 1/\sec\theta$

$\tan\theta = 1/\cot\theta$

$\csc\theta = 1/\sin\theta$

$\sec\theta = 1/\cos\theta$

$\cot\theta = 1/\tan\theta$

Important Trigonometric Identities

Reciprocal Identities

$\sin\theta = 1/\csc\theta$ $\cos\theta = 1/\sec\theta$ $\tan\theta = 1/\cot\theta$

$\cot\theta = 1/\tan\theta$ $\sec\theta = 1/\cos\theta$ $\csc\theta = 1/\sin\theta$

Co function Identities

$\sin\theta = \cos(90 - \theta)$ $\cos\theta = \sin(90 - \theta)$

$\sin\theta = \cos(\pi/2 - \theta)$ $\cos\theta = \sin(\pi/2 - \theta)$

$\tan\theta = \cot(90 - \theta)$ $\cot\theta = \tan(90 - \theta)$

$\tan\theta = \cot(\pi/2 - \theta)$ $\cot\theta = \tan(\pi/2 - \theta)$

$\sec\theta = \csc(90 - \theta)$ $\csc\theta = \sec(90 - \theta)$

$\sec\theta = \csc(\pi/2 - \theta)$ $\csc\theta = \sec(\pi/2 - \theta)$

Quotient Identities

$\tan\theta = \sin\theta/\cos\theta$ $\cot\theta = \cos\theta/\sin\theta$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $\cot^2\theta + 1 = \csc^2\theta$

Introduction to Vectors

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Outline

Definition

Vector analysis is a mathematical tool with which electromagnetic (EM) concepts are most conveniently expressed and best comprehended.

A quantity is called a <u>scalar</u> if it has only magnitude (e.g., mass, temperature, electric potential, population).

A quantity is called a <u>vector</u> if it has both magnitude and direction (e.g., velocity, force, electric field intensity).

The <u>magnitude</u> of a vector $\overline{A}$ is a scalar written as A or $|\overline{A}|$

Unit Vector: Part One

A unit vector $\overline{e}_A$ along $|\overline{A}|$ is defined as a vector whose magnitude is unity (that is,1) and its direction is along

$$\overline{e}_A = \frac{\overline{A}}{|\overline{A}|} = \frac{\overline{A}}{A} \qquad (|\overline{e}_A| = 1)$$

Thus: $\overline{A} = A\overline{e}_A$

which completely specifies $\overline{A}$ in terms of A and its direction $\overline{e}_A$

Unit Vector: Part Two

A unit vector $\bar{e}_A$ along $|A|$ is defined as a vector whose magnitude is unity (that is,1) and its direction is along

$$\bar{e}_A = \frac{\bar{A}}{|\bar{A}|} = \frac{\bar{A}}{A} \quad (|\bar{e}_A| = 1) \qquad \text{Thus:} \quad \bar{A} = A\bar{e}_A$$

which completely specifies $\bar{A}$ in terms of A and its direction $\bar{e}_A$

A vector $\bar{A}$ in Cartesian (or rectangular) coordinates may be represented as

$$(A_x, A_y, A_z) \quad \text{Where:} \quad A_x\bar{e}_x + A_y\bar{e}_y + A_z\bar{e}_z$$

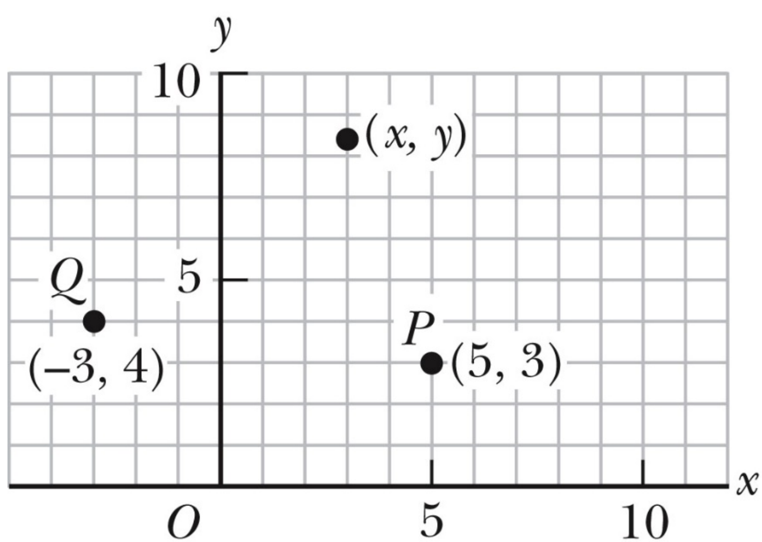
where A_X, A_y, and A_Z are called the components of $\bar{A}$ in the x, y, and z directions, respectively; $\bar{e}_x$, $\bar{e}_y$, and e_z are unit vectors in the x, y and z directions, respectively.

Coordinate Systems

Common coordinate systems are:

- Cartesian
- Polar

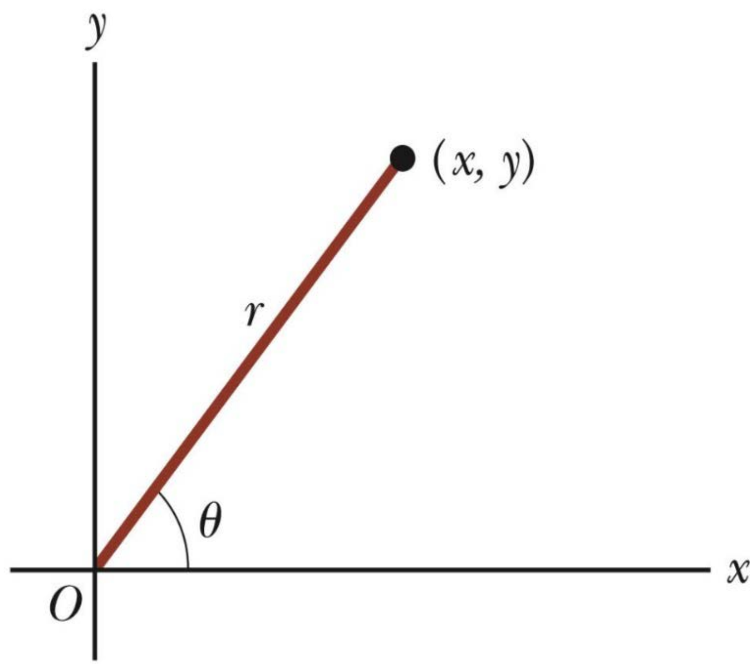
- Also called rectangular coordinate system
- *x*- and *y*- axes intersect at the origin
- Points are labeled (*x*,*y*)



Polar Coordinate System

Origin and reference line are noted

- Point is distance *r* from the origin in the direction of angle *θ*, ccw from reference line
  - The reference line is often the x-axis.
- Points are labeled (*r*,*θ*)



Polar to Cartesian Coordinates

Based on forming a right triangle from r and θ

$$x = r\cos\theta$$

$$y = r\sin\theta$$

If the Cartesian coordinates are known:

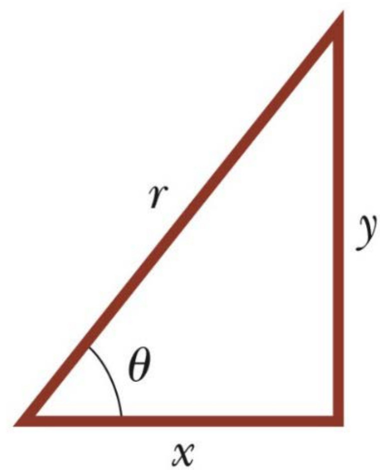
$$\tan\theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$



Vector Addition, Rules

The three basic laws of algebra obeyed by any given vector

A, B, and **C,** are summarized as follows:

Commutative $\overline{A} + \overline{B} = \overline{B} + \overline{A}$ $\quad k\overline{A} = \overline{A}k$

Associative $\overline{A} + (\overline{B} + \overline{C}) = (\overline{A} + \overline{B}) + \overline{C}$ $\quad k(l\overline{A}) = (kl)\overline{A}$

Distributive $k(\overline{A} + \overline{B}) = k\overline{A} + k\overline{B}$

where k and l are scalars

Vector Multiplication: Part One

When two vectors $\overline{A}$ and $\overline{B}$ are multiplied, the result is either a scalar or a vector depending on how they are multiplied. The two types of vector multiplication:

1. Scalar (or dot) product: $\overline{A} \cdot \overline{B}$

2. Vector (or cross) product: $\overline{A} \times \overline{B}$

The dot product of the two vectors $\overline{A}$ and $\overline{B}$ is defined geometrically as the product of the magnitude of $\overline{B}$ and The projection of $\overline{A}$ onto $\overline{B}$ (or vice versa):

$$\overline{A} \cdot \overline{B} = AB\cos\theta_{AB}$$

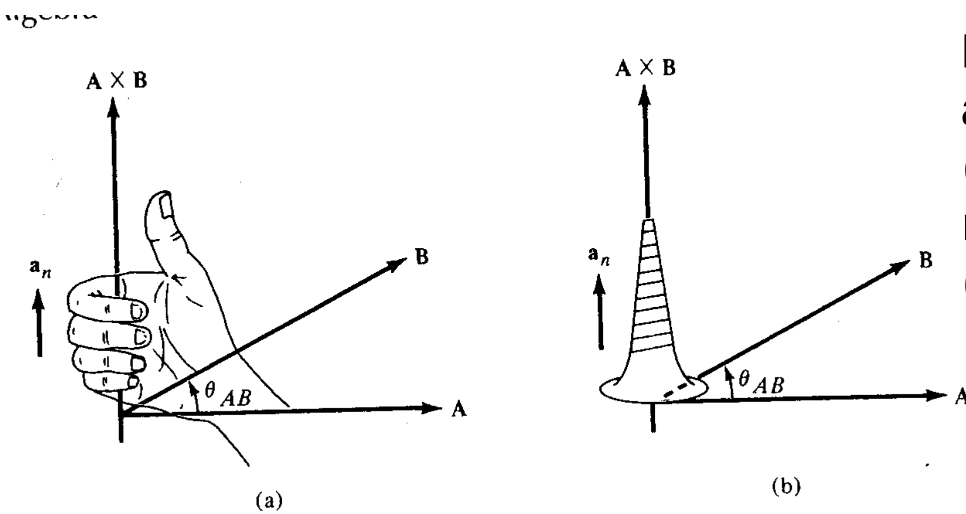
where θ_{AB} is the smaller angle between $\overline{A}$ and $\overline{B}$

Vector Multiplication: Part Two

<u>The cross product</u> of two vectors $\overline{A}$ and $\overline{B}$ is defined as

$$\overline{A} \times \overline{B} = AB \sin\theta_{AB} \overline{e}_n$$

where $\overline{e}_n$ is a unit vector normal to the plane containing $\overline{A}$ and $\overline{B}$. The direction of $\overline{e}_n$ is determined using the right-hand rule or the right-handed screw rule.



Direction of $\overline{e}_n$ and $\overline{A} \times \overline{B}$ using
(a) right-hand rule,
(b) right-handed screw rule

Vector Multiplication: Part Three

Note that the cross product has the following basic properties:

(i) It is <u>not commutative</u>: $\overline{A} \times \overline{B} \neq \overline{B} \times \overline{A}$

It is <u>anticommutative</u>: $\overline{A} \times \overline{B} = -\overline{B} \times \overline{A}$

(ii) It is <u>not associative</u>: $\overline{A} \times (\overline{B} \times \overline{C}) \neq (\overline{A} \times \overline{B}) \times \overline{C}$

(iii) It is <u>distributive</u>: $\overline{A} \times (\overline{B} + \overline{C}) = \overline{A} \times \overline{B} + \overline{A} \times \overline{C}$

(iv) $\overline{A} \times \overline{A} = 0$ $(\sin\theta = 0)$

Chapter Five
Differential Calculus

Developed for Azera Global

By: Joseph D. Fournier B.Sc.E.E., M.Sc.E.E.

Outline

Differential Calculus

The two basic forms of calculus are

- *differential calculus* and
- *integral calculus*.

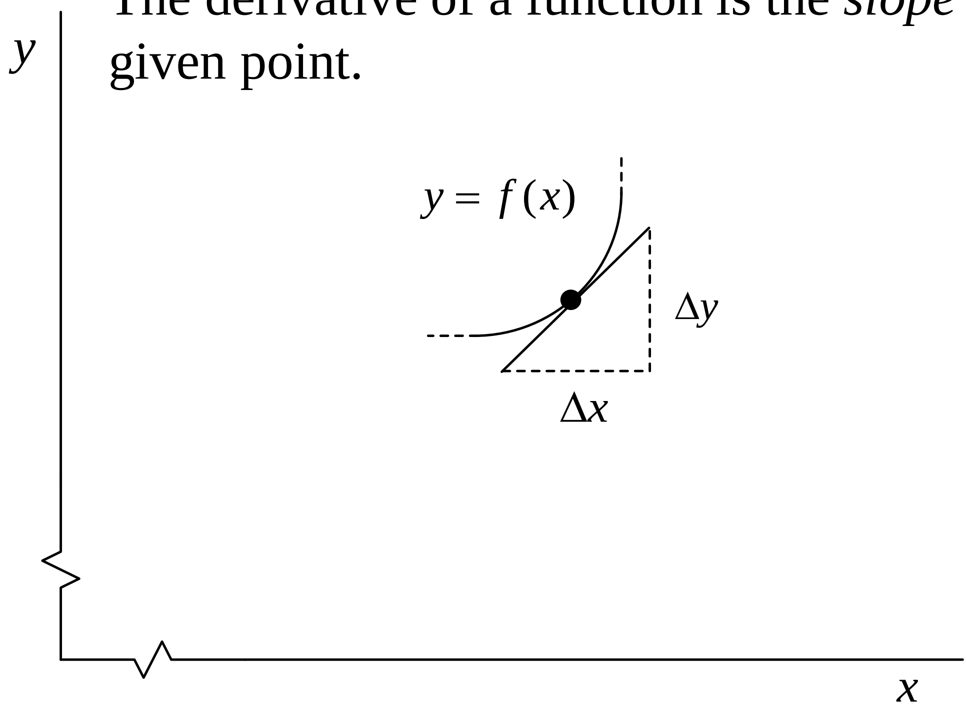
This lecture will be devoted to the former. Integral Calculus will be presented in another lecture.

Differentiation and the Derivative

- The study of calculus begins with the basic definition of a *derivative*. A derivative is obtained through the process of *differentiation,* and the study of all forms of differentiation is collectively referred to as *differential calculus*.
- If we begin with a function and determine its derivative, we arrive at a new function called the *first derivative*.
- If we differentiate the *first derivative*, we arrive at a new function called the *second derivative*, and so on.

Definition of Derivative

The derivative of a function is the *slope* at a given point.



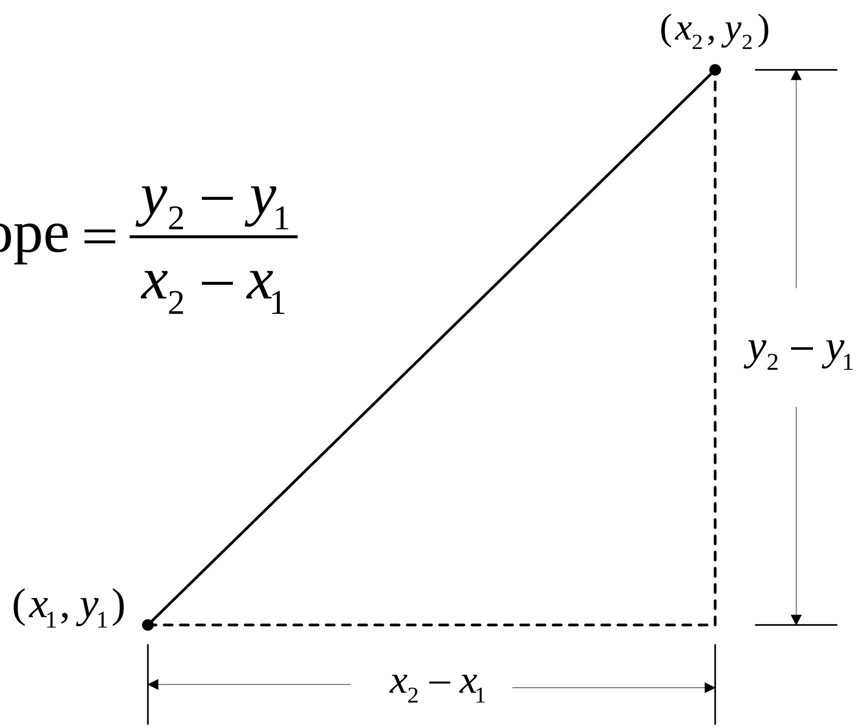
Various Symbols for the Derivative

$$\frac{dy}{dx} \quad \text{or} \quad f\,'(x) \quad \text{or} \quad \frac{df(x)}{dx}$$

Definition: $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Piecewise Linear Segment

$$\frac{dy}{dx} = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$



Example of a Simple Derivative

$$y = x^2$$

$$y + \Delta y = x^2 + 2x\Delta x + (\Delta x)^2$$

$$\Delta y = 2x\Delta x + (\Delta x)^2$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

Chain Rule of Differentiation

$$y = f(u) \qquad u = u(x)$$

$$\frac{dy}{dx} = \frac{df(u)}{du}\frac{du}{dx} = f'(u)\frac{du}{dx}$$

where $$f'(u) = \frac{df(u)}{du}$$

Table of Derivatives: Part One

$f(x)$	$f'(x)$	Derivative Number
$af(x)$	$af'(x)$	D-1
$u(x)+v(x)$	$u'(x)+v'(x)$	D-2
$f(u)$	$f'(u)\frac{du}{dx}=\frac{df(u)}{du}\frac{du}{dx}$	D-3
a	0	D-4
$x^n \qquad (n \neq 0)$	nx^{n-1}	D-5
$u^n \qquad (n \neq 0)$	$nu^{n-1}\frac{du}{dx}$	D-6
uv	$u\frac{dv}{dx}+v\frac{du}{dx}$	D-7
$\frac{u}{v}$	$\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$	D-8
e^u	$e^u\frac{du}{dx}$	D-9

Table of Derivatives:Part Two

a^u	$(\ln a)a^u \dfrac{du}{dx}$	D-10
$\ln u$	$\dfrac{1}{u}\dfrac{du}{dx}$	D-11
$\log_a u$	$(\log_a e)\dfrac{1}{u}\dfrac{du}{dx}$	D-12
$\sin u$	$\cos u\left(\dfrac{du}{dx}\right)$	D-13
$\cos u$	$-\sin u\dfrac{du}{dx}$	D-14
$\tan u$	$\sec^2 u\dfrac{du}{dx}$	D-15
$\sin^{-1} u$	$\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx} \quad \left(-\dfrac{\pi}{2} \le \sin^{-1} u \le \dfrac{\pi}{2}\right)$	D-16
$\cos^{-1} u$	$\dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx} \quad \left(0 \le \cos^{-1} u \le \pi\right)$	D-17
$\tan^{-1} u$	$\dfrac{1}{1+u^2}\dfrac{du}{dx} \quad \left(-\dfrac{\pi}{2} < \tan^{-1} u < \dfrac{\pi}{2}\right)$	D-18

Higher-Order Derivatives

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x) = \frac{df(x)}{dx}$$

$$\frac{d^2y}{dx^2} = f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^3y}{dx^3} = f^{(3)}(x) = \frac{d^3 f(x)}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

Applications: Maxima and Minima

- 1. Determine the derivative.
- 2. Set the derivative to 0 and solve for values that satisfy the equation.
- 3. Determine the second derivative.
  - (a) If second derivative > 0, point is a *minimum*.
  - (b) If second derivative < 0, point is a *maximum*.

Displacement, Velocity, Acceleration

- Displacement $$y$$
- Velocity $$v = \frac{dy}{dt}$$
- Acceleration $$a = \frac{dv}{dt} = \frac{d^2 y}{dt^2}$$

Partial Derivatives and Gradients

Developed for Azera Global

By: Joseph D. Fournier B.Sc.E.E., M.Sc.E.E.

Outline

Definition: Partial Derivative

- the partial derivative of f(x,y) with respect to x and y are

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = (\frac{\partial f}{\partial x})_y = f_x$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = (\frac{\partial f}{\partial y})_x = f_y$$

- second partial derivatives of two-variable function f(x,y)

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \qquad \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

Total Differential

The total differential and total derivative

$$x \to x+\Delta x \text{ and } y \to y+\Delta y \Rightarrow f \to f+\Delta f$$

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x,y)$$

$$= [\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}]\Delta x + [\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}]\Delta y$$

as $\Delta x \to 0$ and $\Delta y \to 0$, the total differential df is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

for n-variable function $f(x_1, x_2, \ldots x_n)$

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots\ldots\ldots\ldots + \frac{\partial f}{\partial x_n}dx_n$$

Exact and Inexact Differentials

If a function can be obtained by directly integrating its total differential, the differential of function f is called exact differential, whereas those that do not are inexact differential.

$$(1)\ df = xdy + (y+1)dx \Rightarrow f(x,y) = xy + x \quad \text{exact differential}$$

$$(2)\ df = xdy + 3ydx$$

$$\Rightarrow \text{function } f(x,y) \text{ doesnot exist} \Rightarrow \text{inexact differential}$$

Properties of exact differentials:

$$A(x,y)dx + B(x,y)dy = df \Rightarrow \frac{\partial f}{\partial x} = A(x,y) \text{ and } \frac{\partial f}{\partial y} = B(x,y)$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial A}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial B}{\partial x} \Rightarrow \frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$$

Properties: Part One

$$x = x(y,z) \Rightarrow dx = (\frac{\partial x}{\partial y})_z dy + (\frac{\partial x}{\partial z})_y dz$$

$$y = y(x,z) \Rightarrow dy = (\frac{\partial y}{\partial x})_z dx + (\frac{\partial y}{\partial z})_x dz$$

$$z = z(x,y) \Rightarrow dz = (\frac{\partial z}{\partial x})_y dx + (\frac{\partial z}{\partial y})_x dy$$

$$dx = (\frac{\partial x}{\partial y})_z (\frac{\partial y}{\partial x})_z dx + [(\frac{\partial x}{\partial y})_z (\frac{\partial y}{\partial z})_x + (\frac{\partial x}{\partial z})_y]dz$$

if z is a constant $\Rightarrow dz = 0$

$$(\frac{\partial x}{\partial y})_z = (\frac{\partial y}{\partial x})_z^{-1} \text{ reciprocity relation}$$

if x is a constant $\Rightarrow dx = 0$

$$(\frac{\partial y}{\partial z})_x (\frac{\partial z}{\partial x})_y (\frac{\partial x}{\partial y})_z = -1 \text{ cyclic relation}$$

Properties: Part Two

The chain rule

$$\text{for } f = f(x,y) \text{ and } x = x(u),\ y = y(u)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \Rightarrow \frac{df}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du}$$

$$\text{for many variables } f(x_1, x_2, \ldots., x_n) \text{ and } x_i = x_i(u)$$

$$\Rightarrow \frac{df}{du} = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{dx_i}{du} = \frac{\partial f}{\partial x_1}\frac{dx_1}{du} + \frac{\partial f}{\partial x_2}\frac{dx_2}{du} + \ldots\ldots.. + \frac{\partial f}{\partial x_n}\frac{dx_n}{du}$$

Partial Differentiation of Integrals

$$F(x,t) = \int f(x,t)dt \Rightarrow \frac{\partial F(x,t)}{\partial x} = f(x,t)$$

$$\Rightarrow \frac{\partial^2 F(x,t)}{\partial t \partial x} = \frac{\partial^2 F(x,t)}{\partial x \partial t} \Rightarrow \frac{\partial}{\partial t}[\frac{\partial F(x,t)}{\partial x}] = \frac{\partial}{\partial x}[\frac{\partial F(x,t)}{\partial t}] = \frac{\partial f(x,t)}{\partial x}$$

$$\Rightarrow \int \frac{\partial}{\partial t}[\frac{\partial F(x,t)}{\partial x}]dt = \int \frac{\partial}{\partial x} f(x,t)dt \Rightarrow \frac{\partial F(x,t)}{\partial x} = \int \frac{\partial f(x,t)}{\partial x}dt$$

Directional Derivatives: Part One

- Recall that, if z = f(x, y), then the partial derivatives f_x and f_y are defined as:

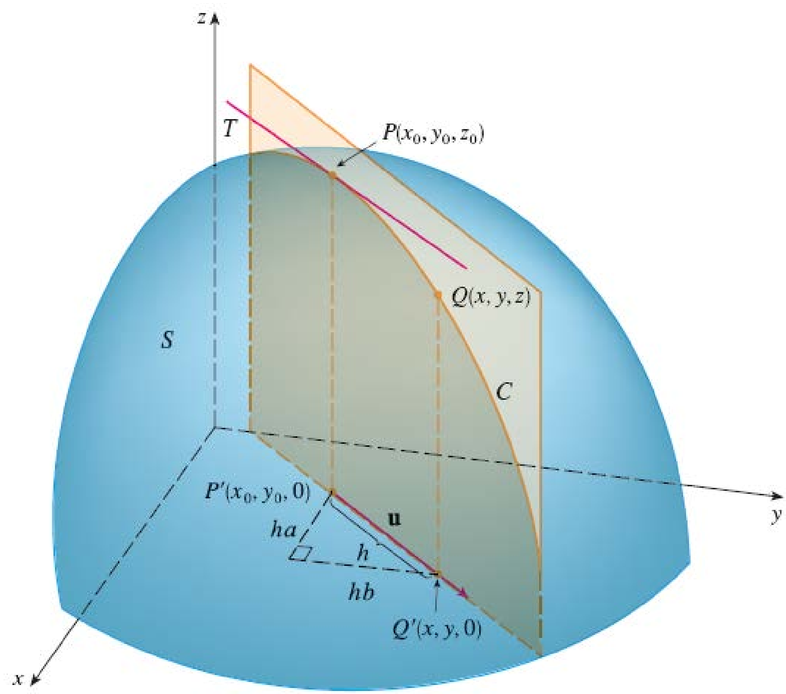
$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Directional Derivatives: Part Two

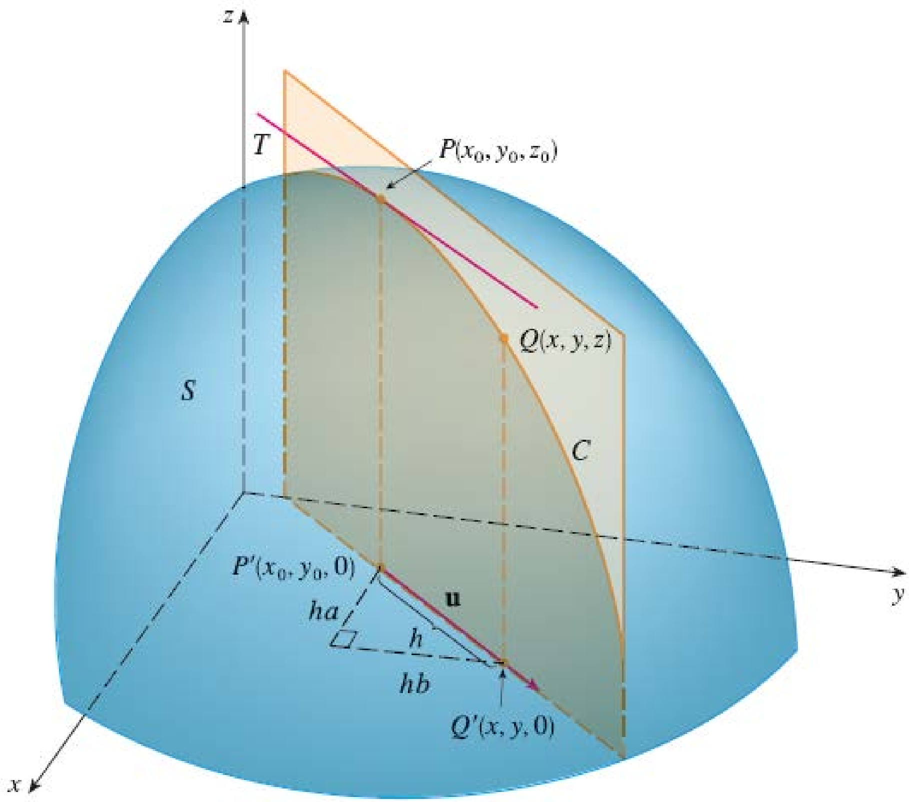
- Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$.

- To do this, we consider the surface S with equation $z = f(x, y)$ [the graph of f] and we let $z_0 = f(x_0, y_0)$.
- Then, the point $P(x_0, y_0, z_0)$ lies on S.



Directional Derivatives: Part Three

- The vertical plane that passes through *P* in the direction of **u** intersects *S* in a curve *C*.
- The slope of the tangent line *T* to *C* at the point *P* is the rate of change of *z* in the direction of **u**.



Directional Derivatives: Part Four

Now, let:

$Q(x, y, z)$ be another point on C.

P', Q' be the projections of P, Q on the xy-plane.

Then the vector $\overrightarrow{P'Q'}$ is parallel to **U**.

So: $\overrightarrow{P'Q'} = h\mathbf{u}$

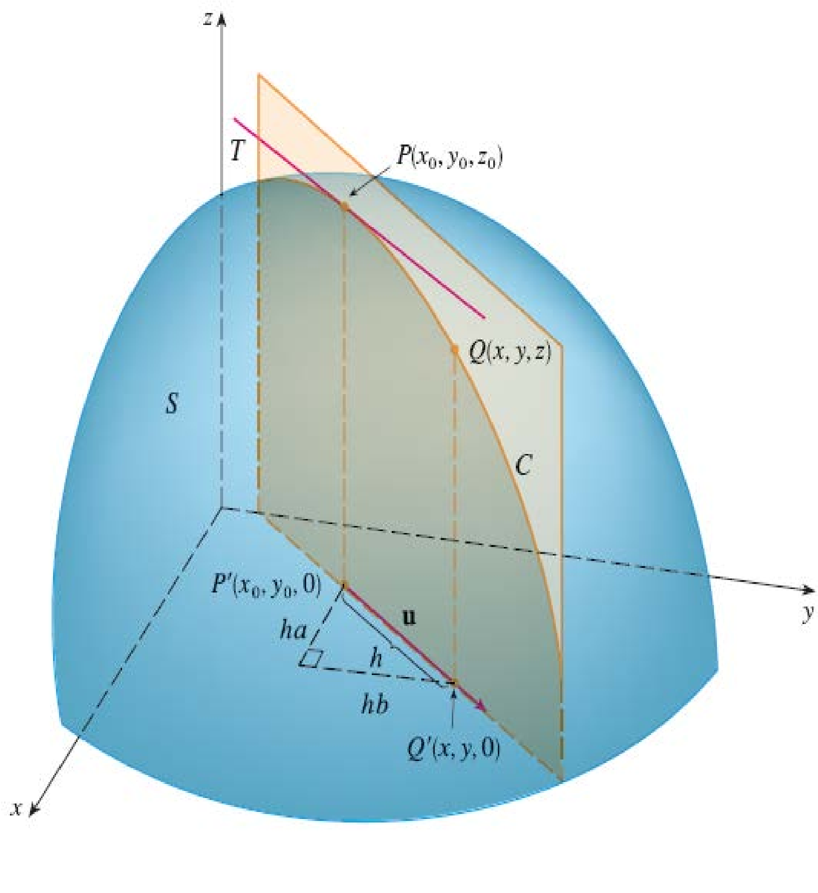
$$= \langle ha, hb \rangle$$

For some scaler h.

Therefore:

$$x - x_0 = ha$$

$$y - y_0 = hb$$



Directional Derivatives: Part Five

From: $x - x_0 = ha$

$y - y_0 = hb$

Then:

$$\frac{\Delta z}{h} = \frac{z - z_0}{h}$$

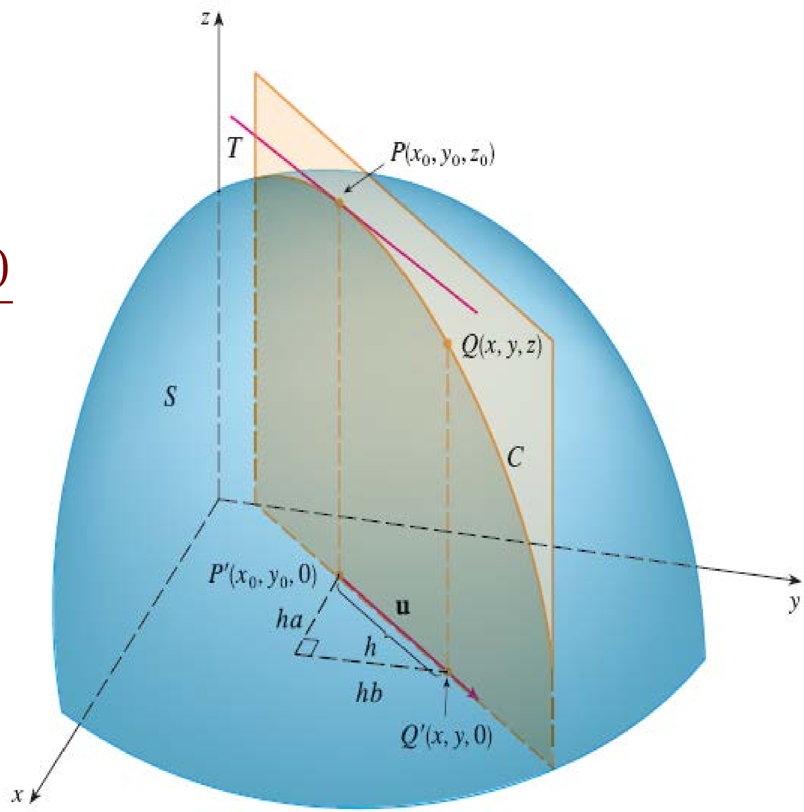
$$= \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

In the limit as $h \to 0$, we obtain the rate of change of z in the direction of **U.**

This is called the directional derivative of f in the direction of **U.**

$$D_{\mathbf{u}} f(x_0, y_0)$$

$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$



Directional Derivatives: Part Six

If we define a function *g* of the single variable *h* by

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

If we define a function *g* of the single variable *h* by:

$$g(h) = f(x_0 + ha, y_0 + hb)$$

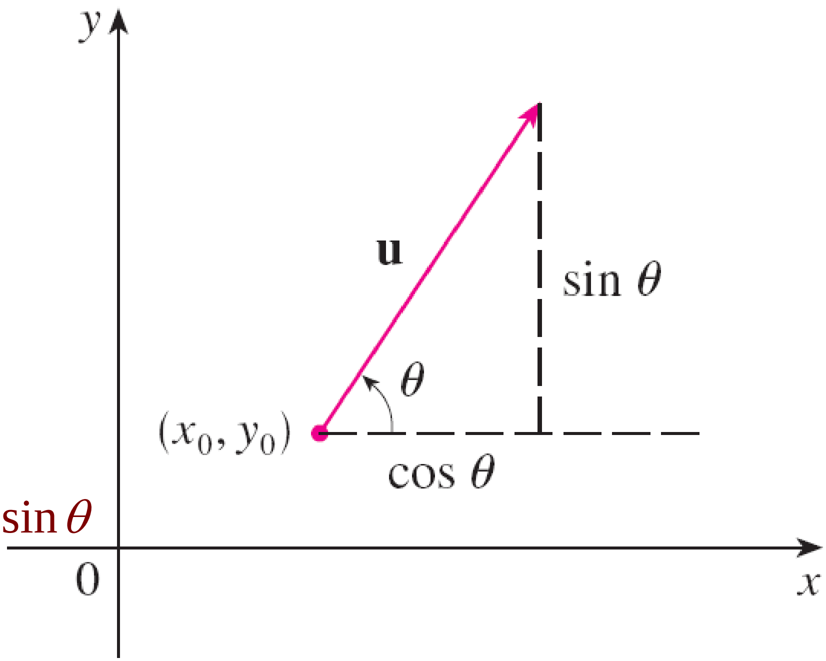
then, by the definition of a derivative, we have the following equation.

$$\begin{aligned} & g'(0) \\ & = \lim_{h \to 0} \frac{g(h) - g(0)}{h} \\ & = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ & = D_{\mathbf{u}} f(x_0, y_0) \end{aligned}$$

Directional Derivatives: Part Seven

Suppose the unit vector **u** makes an angle θ with the positive x-axis, as shown. Then, we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the directional derivative becomes:

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$



Notice that the directional derivative can be written as the dot product of two vectors:

$$
\begin{aligned}
D_{\mathbf{u}} f(x,y) &= f_x(x,y)\,a + f_y(x,y)\,b \\
&= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle \\
&= \langle f_x(x,y), f_y(x,y)\rangle \cdot \mathbf{u}
\end{aligned}
$$

The Gradient: Part One

The first vector in that dot product occurs not only in computing directional derivatives but in many other contexts as well. This directional derivative is called the Gradient of *f.*

The Gradient of *f* is written as: ∇f which is read as "del *f*"

If *f* is a function of two variables *x* and *y* then the gradient of *f(x,y)* is defined as:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{j}$$

We can rewrite the expression for the directional derivative as:

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

This expresses the directional derivative in the direction of **u** as the scalar projection of the gradient vector onto **u**.

The Gradient: Part Two

For functions of three variables, we can define directional derivatives in a similar manner.

The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is:

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

Using vector notation we can rewrite the directional derivative as:

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where:

- $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$
- $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$

The Gradient: Part Three

For a function *f* of three variables, the gradient vector, denoted by ∇f or grad *f*, is:

$$\nabla f(x, y, z)$$

$$= \langle f_x(x, y, z), f_y(x, y, z,), f_z(x, y, z) \rangle$$

And is written as: $\nabla f = \langle f_x, f_y, f_z \rangle$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

The directional derivative can be rewritten as:

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is: $|\nabla f(\mathbf{x})|$ and it occurs when **u** has the same direction as the gradient vector $\nabla f(\mathbf{x})$

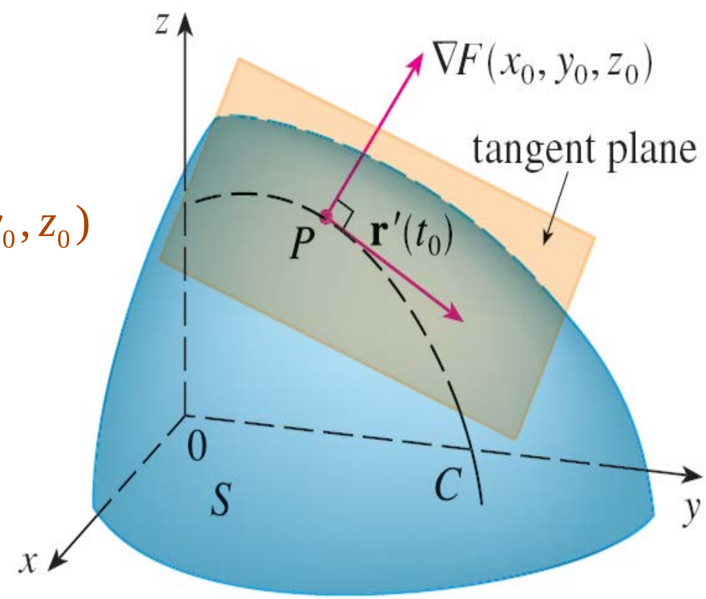
Tangent Plane

Suppose S is a surface with equation $F(x, y, z)$ that is, it is a level surface of a function F of three variables.

Then, let $P(x_0, y_0, z_0)$ be a point on S.

Then, let C be any curve that lies on the surface S and passes through the point P.

The curve C is described by a continuous vector function $\mathbf{r}(t) = <x(t), y(t), z(t)>$
The gradient vector at P $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ and to any curve C on S that passes through P.
Thus the direction of the normal line is given by the gradient vector. $\nabla F(x_0, y_0, z_0)$



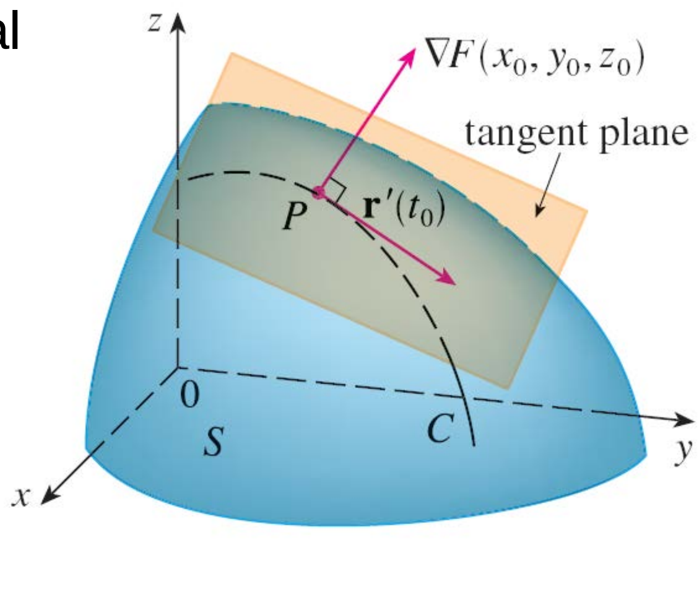
Summary of Gradient

We now summarize the ways in which the gradient vector is significant.

For a function f of three variables and a point $P(x_0, y_0, z_0)$ in its domain we know that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of f.

On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface S of f through P.

So, it seems reasonable that, if we move in the perpendicular direction, we get the maximum increase.



Chapter Six:
Integral Calculus

Developed for Azera Global

By: Joseph D. Fournier B.Sc.E.E., M.Sc.E.E.

Outline

Integral Calculus

- The basic concepts of *differential calculus* were covered in the preceding presentation. This presentation will be devoted to *integral calculus*, which is the other broad area of calculus.

Anti-Derivatives

An anti-derivative of a function $f(x)$ is a new function $F(x)$ such that

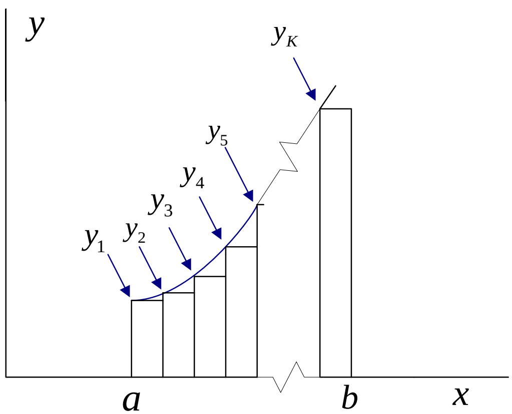
$$\frac{dF(x)}{dx} = f(x)$$

Indefinite and Definite Integrals

Indefinite $$\int f(x)dx$$

Definite $$\int_{x_1}^{x_2} f(x)dx$$

Definite Integral/ Area Under the Curve



$$\text{Approximate Area} = \sum_k y_k \Delta x$$

Exact Area as Definite Integral

$$\int_a^b y\,dx = \lim_{\Delta x \to dx} \sum_k y_k \Delta x$$

Definite Integral with Variable Upper Limit

$$\int_a^x y\,dx$$

More “proper” form with “dummy” variable

$$\int_a^x y(u)\,du$$

Guidelines

- If *y* is a non-zero constant, integral is either increasing or decreasing linearly.
- If segment is triangular, integral is increasing or decreasing as a parabola.
- If $y=0$, integral remains at previous level.
- Integral moves up or down from previous level; i.e., no sudden jumps.
- Beginning and end points are good reference levels.

Tabulation of Integrals

$$F(x) = \int f(x)dx$$

$$I = \int_a^b f(x)dx$$

$$I = F(x)\Big]_a^b = F(b) - F(a)$$

Common Integrals: Part One

$f(x)$	$F(x)=\int f(x)dx$	Integral Number
$af(x)$	$aF(x)$	I-1
$u(x)+v(x)$	$\int u(x)dx+\int v(x)dx$	I-2
a	ax	I-3
$x^n \quad (n\neq -1)$	$\frac{x^{n+1}}{n+1}$	I-4
e^{ax}	$\frac{e^{ax}}{a}$	I-5
$\frac{1}{x}$	$\ln x$	I-6
$\sin ax$	$-\frac{1}{a}\cos ax$	I-7
$\cos ax$	$\frac{1}{a}\sin ax$	I-8
$\sin^2 ax$	$\frac{1}{2}x-\frac{1}{4a}\sin 2ax$	I-9

Common Integrals: Part Two

$\cos^2 ax$	$\frac{1}{2}x+\frac{1}{4a}\sin 2ax$	I-10
$x\sin ax$	$\frac{1}{a^2}\sin ax-\frac{x}{a}\cos ax$	I-11
$x\cos ax$	$\frac{1}{a^2}\cos ax+\frac{x}{a}\sin ax$	I-12
$\sin ax\cos ax$	$\frac{1}{2a}\sin^2 ax$	I-13
$\sin ax\cos bx$ for $a^2\neq b^2$	$-\frac{\cos(a-b)x}{2(a-b)}-\frac{\cos(a+b)x}{2(a+b)}$	I-14
xe^{ax}	$\frac{e^{ax}}{a^2}\left(ax-1\right)$	I-15
$\ln x$	$x\left(\ln x-1\right)$	I-16
$\frac{1}{ax^2+b}$	$\frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{a}{b}}\right)$	I-17

Displacement, Velocity, Acceleration

$$a = a(t) = \text{acceleration in meters/second}^2 \ (\text{m/s}^2)$$

$$v = v(t) = \text{velocity in meters/second (m/s)}$$

$$y = y(t) = \text{displacement in meters } (\text{m})$$

$$\frac{dv}{dt} = a(t) \quad dv = \left(\frac{dv}{dt}\right)dt = a(t)dt \quad \int dv = \int a(t)dt \quad v = \int a(t)dt + C_1$$

$$\int dv = v \qquad dy = \left(\frac{dy}{dt}\right)dt = v(t)dt$$

$$\frac{dy}{dt} = v(t) \qquad y = \int v(t)dt + C_2$$